The Extended Supersymmetrization of the Multicomponent Kadomtsev–Petviashvilli Hierarchy

by
Ziemowit Popowicz

Institute of Theoretical Physics, University of Wrocław
Pl. M. Borna 9 50 - 205 Wrocław Poland

Abstract

We describe three different approaches to the extended (N=2) supersymmetrization of the multicomponent KP hierarchy. In the first one we utilize only superfermions while in the second only superbosons and in the third superbosons as well as superfermions. It is shown that many soliton equations can be embedded in the supersymmetry theory by using the first approach even if we do not change these equations in the bosonic limit of the supersymmetry. In the second or third approach we obtain a generalization of the soliton equations in the bosonic limit which remains in the class of the usual commuting functions. As the byproduct of our analysis we prove that for the first procedure the bosonic part of the one–component supersymmetric KP hierarchy coincides with the usual classical two–component KP hierarchy.

1 Introduction.

Integrable Hamiltonian systems occupy an important place in diverse branches of theoretical physics as exactly solvable models of fundamental physical phenomena ranging from nonlinear hydrodynamics to string theory [1,2,3]. The general Kadomtsev - Petviashvilli (KP) system [4,5] is 1+1 dimensional integrable model containing an infinite number of fields. In the Sato approach [6,7,8], the KP hierarchy is described by the isospectral deformations of the eigenvalue problem $L\Psi = \lambda\psi$ for the pseudodifferential operator $L = \partial + U_2\partial^{-1} + U_3\partial^{-2}$ which is given by

$$L_{t_n} = [B_n, L]\,, \tag{1}$$

where $n = 2, 3, ...$ and B_n is the differential part of the microdifferential operator L^n. If we require that L satisfies the additional condition that $L^n = B_n, n \geq 2$ then the hierarchy of equation given by Eq. (1) are reduced to the hierarchy of (1+1) dimensional integrable systems called the n-reduced KP hierarchy. For example, the Korteweg - de Vries equation and the Boussinesq equation belong to the two-reduced and three-reduced KP hierarchies, respectively.

On the other side, a new type of reduction have been proposed recently in a series of articles, which leads many (2+1) dimensional integrable systems to (1+1) dimensional integrable systems [9–15]. For example, by assuming that L satisfies the constraints

$$L^n = B_n + q\partial^{-1}r, \tag{2}$$

we can obtain the so called k-constrained KP hierarchy. Interestingly, the one - constrained KP hierarchy coincide with the AKNS hierarchy, and the two - constrained KP hierarchy coincides with the Yajima - Oikawa [16] hierarchy. The k-constrained KP hierarchy was shown to possess Lax pairs, recursion operators and the bi-Hamiltonian structures [15].

However this classification does not exhaust the known generalizations of the KP hierarchies. In this paper we consider two different generalizations of the KP hierarchies. In the next section we describe the so called multicomponent KP hierarchy and in the next chapters we consider the extended supersymmetrization of the multicomponent KP hierarchies.

The idea to use the extended supersymmetry (SUSY) for the generalization of the soliton equations appeared almost in parallel to the usage of

the SUSY in the quantum field theory [17,18]. The main idea of SUSY is to treat bosons and fermions operators equally. The first results, concerned the construction of classical field theories with fermionic and bosonic fields depending on time and one space variable, can be found in [19–22]. In many cases, the addition of fermion fields does not guarantee that the final theory becomes the SUSY invariant. Therefore this method was named the fermionic extension in order to distinguish it from the fully SUSY way.

We have at the moment many different procedures [23 -48] of the supersymmetrization of the soliton equations. From the soliton point of view we can distinguish two different recipes. In the first we add to the theory the new anticommuting Grassmann valued functions only while in the second case we also add the new commuting functions. Interestingly enough it appeared that during the supersymmetrizations, some typical SUSY effects (compared to the classical theory) occurred. We mention few of them ; the nonuniqueness of the roots for the SUSY Lax operator [36,40], the lack of the bosonic reduction to the classical equations [35] and the nonexistence of the extended SUSY extension of the $SL(2, C)$ Kac- Moody algebra [30]. These effect strongly relies on the descriptions of the generalized classical systems of equations which we would like to supersymmetrize. In the classical case the AKNS hierarchy is connected with the one–component KP hierarchy and in the SUSY case it is tempting to use similar arguments and try to construct the SUSY version of AKNS hierarchy. However we are not able to show that the supersymmetric extension constitutes the SUSY bihamiltonian system. Therefore, we do not consider here the problem of the extended supersymmetrization of the AKNS hierarchy. We show that the lack of the existence of the bihamiltonian structure in our approach is closely connected with the nonexistence of the extended SUSY $SL(2, C)$ Kac-Moody algebra.

From this classification of the supersymmetrization methods one can infer that the second approach is more important then the first because we extend our knowledge on the new commuting functions. However, it is not completely true. As we show in this paper, it is possible to carry out the supersymmetrization of the one–component KP hierarchy in two different ways. We show that despite of using the superfermions in the first approach for the supersymmetrization of the one–component KP hierarchy, the bosonic sector coincides with the usual classical two–component KP hierarchy. Interestingly the bosonic part of the SUSY Lax pair of the one–component KP hierarchy is matrix valued operator in contrast to the scalar Lax operator in

the classical case.

The paper is organized as follows. In the first section we describe the multicomponent KP hierarchy. The second contains the introduction to the supersymmetrization of this hierarchy which is developed in the next chapters. More precisely in the third section we describe the superfermionic approach while in the fourth the superbosonic. We use superfermions as well as superbosons in the fifth chapter in the supersymmetrization of our multicomponent KP hierarchy in order to demonstrate the third (mixed) possibilities. The last section contains concluding remarks.

All calculations presented in this paper have been obtained by the extensive applications of the symbolic computation language REDUCE.

2 Multicomponent KP hierarchy.

The multicomponent KP hierarchy have been introduced by Sidorenko and Strampp [14] which is a straightforward generalization of the scalar case. This is a hierarchy associated with the following Lax operator

$$L_n = \partial^n + u_{n-2}\partial^{n-2} + \ldots + u_0 + \sum_{u=1}^{m} q_i\partial^{-1}r_i \, , \tag{3}$$

The corresponding flows could be constructed by means of fractional power method [5]. For n=1, one has multicomponent AKNS hierarchy, which includes coupled Nonlinear Schrödinger [49] equation as an example. For the case n=2 and n=3 one has the multicomponent Yajima - Oikawa [16] and Melnikov hierarchy [50] respectively. We first consider multicomponent AKNS hierarchy which is given by

$$L = \partial + \sum_{i=1}^{n} q_i\partial^{-1}r_i \, , \tag{4}$$

and the flows are

$$L_{t_k} = \left[\left(L^k \right)_+ , L \right] \, . \tag{5}$$

The bi-hamiltonian structure of these equations have been widely discussed in the literature recently [14,51,52] and it has the following representation

$$q_{t_k} = B^0 \frac{\delta H_{k+1}}{\delta q} = B^1 \frac{\delta H_k}{\delta q} \, , \tag{6}$$

where $q = (q_1, q_2, ...q_n, r_1, r_2, \dots r_n)$ and

$$B^0 = \begin{pmatrix} O & I \\ -I & O \end{pmatrix} , \tag{7}$$

where I is $m \times m$ identity matrix. B^1 is in the form [52]

$$B^1 = \begin{pmatrix} B^1_{11}, & B^1_{12} \\ \\ B^1_{21}, & B^1_{22} \end{pmatrix} , \tag{8}$$

where $B^1_{n,k}(n,k = 1,2)$ are $m \times m$ matrices with the entries

$$B^1_{11} = \left\{ q_i \partial^{-1} q_j + q_j \partial^{-1} q_i \right\} , B^1_{12} = \left\{ \left(\partial - \sum_{s=1}^{m} q_s \partial^{-1} r_s \right) \delta_{ij} - q_i \partial^{-1} r_j \right\} \tag{9}$$

$$(B^1_{12})^* = -B^1_{21} , \qquad\qquad B^1_{22} = \left\{ r_i \partial^{-1} r_j + r_j \partial^{-1} r_i \right\} \tag{10}$$

and $*$ denotes the hermitean conjugation. In the special case $m = 1$ we obtain

$$B^1 = \begin{pmatrix} 2q\partial^{-1}q, & \partial - 2q\partial^{-1}r \\ \\ \partial - 2r\partial^{-1}q, & 2r\partial^{-1}r \end{pmatrix} , \tag{11}$$

Interestingly, this Hamiltonian operator could be considered as the outcome of the Dirac reduction of the hamiltonian operator connected with the $SL(2,C)$ Kac-Moody algebra [39].

The Hamiltonians H_k may be computed from

$$H_k = \frac{1}{k} Res \left(L^k \right) , \tag{12}$$

in which Res denotes the coefficient standing in ∂^{-1} term.

For the subsequent discussion let us explicitly presents the equations (6) for the two–component KP hierarchy in the two particular cases.

For $k = 2$ these equations are in the form

$$q_{i_t} = q_{ixx} + 2q_i \sum_{i=1}^{m} q_s r_s , \tag{13}$$

$$r_{i_t} = -r_{ixx} - 2r_i \sum_{i=1}^{m} q_s r_s \,. \tag{14}$$

This is a vector generalization of the Nonlinear Schrödinger equation considered first time in [49].
For $k = 3$

$$q_{i_t} = q_{ixxx} + 3q_i \sum_{s=1}^{m} q_{sx} r_s + 3q_{ix} \sum_{s=1}^{m} q_s r_s \,, \tag{15}$$

$$r_{i_t} = r_{ixxx} + 3r_i \sum_{s=1}^{m} q_s r_{sx} + 3r_{ix} \sum_{s=1}^{m} q_s r_s \,. \tag{16}$$

These equations could be further restricted to the known soliton equation. Indeed, assuming that m=1 we obtain that equations (12) - (13) reduce to the usual Nonlinear Schrödinger equation while the eqs. (14)-(15) for $q = r$ to the modified Korteweg - de Vries equation or for $r = 1$ to the Korteweg - de Vries equation.

3 The extended supersymmetrization of the multicomponent KP hierarchy.

The basic objects in the supersymmetric analysis are the superfield and the supersymmetric derivative. We will deal with the so called extended $N = 2$ supersymmetry for which superfields are the superfermions or the superbosons depending, in addition to x and t, upon two anticommuting variables, θ_1 and θ_2, $(\theta_2\theta_1 = -\theta_1\theta_2, \theta_1^2 = \theta_2^2 = 0)$. Their Taylor expansion with respect to the θ is

$$\phi(x, \theta_1, \theta_2) = w(x) + \theta_1\zeta_1(x) + \theta_2\zeta_2(x) + \theta_2\theta_1 u(x), \tag{17}$$

where the fields w, u, are to be interpreted as the boson (fermion) fields for the superboson (superfermion) field ζ_1, ζ_2, as the fermions (bosons) for the for the superboson (superfermion) respectively. The superderivatives are defined as

$$\mathcal{D}_1 = \partial_{\theta_1} + \theta_1\partial \,, \qquad \mathcal{D}_2 = \partial_{\theta_2} + \theta_2\partial, \tag{18}$$

with the properties

$$\mathcal{D}_2\mathcal{D}_1 + \mathcal{D}_1\mathcal{D}_2 = 0 \,, \qquad \mathcal{D}_1^2 = \mathcal{D}_2^2 = 0. \tag{19}$$

Below we shall use the following notation: $(\mathcal{D}_i F)$ denotes the outcome of the action of the superderivative on the superfield F, while $\mathcal{D}_i F$ denotes the action itself of the superderivative on the superfield F.

The principal problem in the supersymmetrization of the soliton equations could be formulated as follows : if we know the evolution equation for the classical function u and its (bi) hamiltonian structure or its Lax pair, how is its possible to obtain the evolution equation on the supermultiplet Φ which contains the classical function u? This problem has its own history and at the moment we have no an unique solution. We can distinguish three different methods of the supersymmetrization, as for example the algebraic, geometric and direct method.

In the first two cases we are looking for the symmetry group of the given equation and then we replace this group by the corresponding SUSY group. As a final product we are able to obtain the SUSY generalization of the given equation. The classification as the algebraical or geometrical approach is connected with the kind of symmetry which appears on the classical level. For example, if our classical equation could be described in terms of the geometrical object then the simple exchange of the classical symmetry group of this object onto SUSY partner justify the name geometric. In the algebraic case, we are looking for the symmetry group of this equation without any reference to its geometrical origin. This strategy could be applied to the so called hidden symmetry as for example in the case of the Toda lattice [54].

These methods have both advantages and disadvantages. For example, sometimes we obtain the fermionic extensions of the given equations only [44,54]. In the case of the extended supersymmetric Korteweg-de Vries equation we have three different fully SUSY extensions, however only one of them fits to these two classifications [26–28,31].

It seems that the most difficult problem in these approaches is the explanation why a'priori SUSY extension of the classical system of equation should be connected with the SUSY extension of the classical symmetry of these equations. By these reasons we prefer to use the direct approach in which we simply replace all objects which appear in the evolution equation by all possible supermultiplets and superderivatives in such a way that to conserve the gradations of the equation. This is highly non unique procedure and we obtain a lot of different possibilities. However this arbitrariness can be restricted if we additionally investigate its super-bi-hamiltonian structure or try to find its supersymmetric Lax pair. In many cases this manner bring

the success [34,35,40–48]. In the next we utilize this way.

Let us now start our considerations of trying to find the Lax operator for the multicomponent SUSY KP hierarchy. The direct method suggests to assume that L depends on the vectors supermultiplets F, G its supersymmetric derivatives and on the derivative and superderivatives in such a way that finally it has the gradation 1. Therefore we postulate that the Lax pair is an operator in the form

$$L = L\left(\partial, D_1, D_2, F, G\right). \tag{20}$$

In order to specify this form we have to assume the gradations of the supermultiplets F and G. However we quickly recognize that we encounter three different possibilities of the gradations of F, G:

1.) All F, G are superfermions with the gradation 1/2,

2.) All F, G are superbosons with the following gradation: F has 0 while G has 1 (or symmetrically).

3.) The mixture of both previous possibilities in other words some of the F and G are superbosons and the rest are superfermions.

In the next sections we investigate in more details these possibilities.

4 The superfermionic approach.

We now assume that the components of the vectors supermultiplets F and G are superfermions which could be written down as

$$F_i = \zeta_i^1 + \theta_1 f_i^1 + \theta_2 f_i^2 + \theta_2\theta_1\zeta_i^2, \tag{21}$$

$$G_i = \eta_i^1 + \theta_1 g_i^1 + \theta_2 g_i^2 + \theta_2\theta_1\eta_i^2, \tag{22}$$

where f_j^i, g_i^k are usual classical functions while ζ_j^k, η_i^k are Grassmannian valued functions. The Lax operator we choose in such a way that contains all possible combinations of "variables" in the (20) in such manner that each term has gradation 1. Then using the symbolic language REDUCE we verified that the following operator

$$L = \partial + \sum_{i=1}^{k} F_i \cdot \partial^{-1} \cdot D_1 \cdot D_2 \cdot G_i, \tag{23}$$

generate extended supersymmetric multicomponent KP hierarchy. Indeed, its second flow is

$$F_{i_t} = F_{ixx} + 2\sum_{s=1}^{k} F_s\left(\mathcal{D}_1\mathcal{D}_2 G_s F_i\right) - F_i\left(\sum_{s=1}^{k} F_s G_s\right)^2 , \tag{24}$$

$$G_{i_t} = G_{ixx} + 2\sum_{s=1}^{k} G_s\left(\mathcal{D}_1\mathcal{D}_2 G_i F_s\right) + G_i\left(\sum_{s=1}^{k} F_s G_s\right)^2 , \tag{25}$$

while the third is

$$\begin{aligned} F_{i_t} = \quad & F_{ixxx} + 3\sum_{j=1}^{k}\Big\{\left(\mathcal{D}_1\mathcal{D}_2 G_j F_{ix}\right) + \left(\mathcal{D}_1\mathcal{D}_2 G_j F_i\right) F_{jx} - \\ & \sum_{l=1}^{k}\left[\left(\mathcal{D}_1 G_j F_l\right)\left(\mathcal{D}_1 G_l F_i\right) F_j + \left(\mathcal{D}_2 G_j F_l\right)\left(\mathcal{D}_2 G_l F_i\right) F_j\right]\Big\} - 3F_{ix}Z , \end{aligned} \tag{26}$$

$$\begin{aligned} G_{i_t} = \quad & G_{ix} + 3\sum_{j=1}^{k}\Big\{\left(\mathcal{D}_1\mathcal{D}_2 G_{ix} F_j\right) + G_j\left(\mathcal{D}_1\mathcal{D}_2 G_i F_j\right) G_{jx} + \\ & + \sum_{l=1}^{k}\left(\mathcal{D}_1 G_i F_j\right)\left(\mathcal{D}_1 G_i F_l\right) G_l + \left(\mathcal{D}_2 G_i\right)\left(\mathcal{D}_2 G_j F_l\right) G_l\Big\} - 3G_{ix}Z , \end{aligned} \tag{27}$$

where

$$Z = \sum_{i,j=1}^{k} F_i G_i F_j G_j . \tag{28}$$

Let us now discuss several particular cases of the equations (24–27). For k=1, equations (24–25) reduces to

$$F_t = F_{xx} + 2F\left(\mathcal{D}_1\mathcal{D}_2 G F\right) , \tag{29}$$

$$G_t = -G_{xx} - 2G\left(\mathcal{D}_1\mathcal{D}_2 G F\right) , \tag{30}$$

In the components, using (21–22), we obtained that eqs. (29–30) are equivalent with

$$\zeta^1_t = \zeta^1_{xx} + 2\zeta^1\left(\eta^1\zeta^2 + f^1 g^2 - f^2 g^1\right) , \tag{31}$$

$$f^1_t = f^1_{xx} - 2\zeta^1\left(g^2\zeta^1 - f^2\eta^2\right)_x + 2f^1\left(\eta^1\zeta^2 + \eta^2\zeta^1 + f^1 g^2 - f^2 g^1\right) , \tag{32}$$

$$f_t^2 = f_{xx}^2 + 2\zeta^1 \left(g^1\zeta^1 - f^1\eta^1\right)_x + 2f^2 \left(\eta^1\zeta^2 + \eta^2\zeta^1 + f^1g^2 - f^2g^1\right), \quad (33)$$

$$\begin{aligned} \zeta_t^2 = \; & \zeta_{xx}^2 - 2\zeta^1 \left(\eta^1\zeta^1\right)_{xx} + 2f^1 \left(g^1\zeta^1 - f^1\eta^1\right)_x + \\ & 2f^2 \left(g^2\zeta^1 - f^2\eta^1\right)_x + 2\zeta^2 \left(\eta^2\zeta^1 + f^1g^2 - f^2g^1\right), \end{aligned} \quad (34)$$

$$\eta_t^1 = -\eta_{xx}^1 - 2\eta^1 \left(\eta^2\zeta^1 + f^1g^2 - f^2g^1\right), \quad (35)$$

$$g_t^1 = -g_{xx}^1 + 2\eta^1 \left(g^2\zeta^1 - f^2\eta^1\right)_x - 2g^1 \left(\eta^1\zeta^2 + \eta^2\zeta^1 + f^1g^2 - f^2g^1\right), \quad (36)$$

$$g_t^2 = -g_{xx}^2 - 2\eta^1 \left(g^1\zeta^1 - f^1\eta^1\right)_x - 2g^2 \left(\eta^1\zeta^2 + \eta^2\zeta^1 + f^1g^2 - f^2g^1\right), \quad (37)$$

$$\begin{aligned} \eta_t^2 = \; & -\eta_{xx}^2 + 2\eta^1 \left(\eta^1\zeta^1\right)_{xx} - 2g^1 \left(g^1\zeta^1 - f^1\eta^1\right)_x - \\ & 2g^2 \left(g^2\zeta^1 - f^2\eta^1\right)_x + 2\eta^2 \left(\eta^1\zeta^2 + f^1g^2 - f^2g^1\right). \end{aligned} \quad (38)$$

As we see, this system of equations can be interpreted as the extended supersymmetric Nonlinear Schrödinger equation which have been widely discussed recently [36-41,48]. The bosonic part (in which all fermions fields vanishes) give us the equations (7) for m=2 with the following identifications

$$f^1 = g_1\,, \quad f^2 = g_2\,, \quad q^1 = -r_2\,, \quad q^2 = r_1. \quad (39)$$

Interestingly, our Lax operator in the bosonic limit for k=1 does not reduce to the scalar Lax pair (4). In our case, it has a matrix form

$$L = \begin{pmatrix} \partial + q_1\partial^{-1}r_1\,, & q_1\partial^{-1}r_2 \\ q_2\partial^{-1}r_1\,, & \partial + q_2\partial^{-1}r_2 \end{pmatrix}. \quad (40)$$

In this way, we have shown that our one–component extended supersymmetric KP hierarchy in the bosonic sector is equivalent with the usual two–component KP hierarchy. Moreover, in this bosonic sector, our equations constitute the bi-Hamiltonian structure given by (6-11), but we are not able to find its supersymmetric bihamiltonian counterparts. This fact probably is connected with the nonexistence of the extended (N=2) supersymmetric SL(2,C) Kac-Moody algebra [30]. This SL(2,C) Kac-Moody algebra plays a crucial meaning in the AKNS approaches and to its bi-Hamiltonian structure as it was shown in the previous section. Moreover, the applications of the direct method to the supersymmetrization of the formula (11) also does

not give us the correct solution, what we have checked using the symbolic computation program REDUCE.

On the other hand our equations are Hamiltonian equations which can be written as

$$\begin{pmatrix} F \\ \\ G \end{pmatrix}_{t_k} = \begin{pmatrix} 0, & I \\ \\ -I, & 0 \end{pmatrix} \begin{pmatrix} \dfrac{\delta H_k}{\delta F} \\ \dfrac{\delta H_k}{\delta G} \end{pmatrix}, \tag{41}$$

where $F = (F_1, F_2, ...F_k)^t$, $G = (G_1, G_2, ...G_k)^t$ and I is a $k \times k$ identity matrix. The Hamiltonians H_k can be computed by using the formula (12) in which now theRes denotes the coefficient standing in $\partial^{-1} D_1 D_2$ term.

Let us now discuss the equations (25-26) for k=1. In this case they reduce to

$$F_t = F_{xxx} + 3\left[(\mathcal{D}_1\mathcal{D}_2 G F_x) F + (\mathcal{D}_1\mathcal{D}_2 G F_x) F_x\right], \tag{42}$$

$$G_t = G_{xxx} + 3\left[(\mathcal{D}_1\mathcal{D}_2 G_x F) G + (\mathcal{D}_1\mathcal{D}_2 G F_x) G_x\right], \tag{43}$$

with the following bosonic sector

$$f_{1t} = f_{1xxx} - 3g_1 (f_2 f_1)_x + 3g_2 \left(f_1^2\right)_x, \tag{44}$$

$$f_{2t} = f_{2xxx} + 3g_2 (f_1 f_2)_x - 3g_1 \left(f_2^2\right)_x, \tag{45}$$

$$g_{1t} = g_{1xxx} + 3f_1 (g_1 g_2)_x - 3f_2 \left(g_1^2\right)_x, \tag{46}$$

$$g_{2t} = g_{2xxx} - 3f_2 (g_1 g_2)_x + 3f_1 \left(g_2^2\right)_x. \tag{47}$$

This system of equation can be considered as the vector generalizations of the Modified Korteweg - de Vries equation. Now we can investigate different reductions of the eqs. (44-47) to much simpler equations. For example, by assuming that

$$g_1 = f_1 = f_2, \qquad\qquad g_2 = 0, \tag{48}$$

we obtain the usual Modified Korteweg - de Vries equation.

To finish this section let us notice that this superfermionic manner discussed in this section allows us to obtain some extension of the usual system of equations by incorporating anticommuting functions but we do not change the usual multicomponent K-P hierarchy. We show in the next sections that superbosonic or mixed ways of supersymmetrizations generalize our usual multicomponent K-P hierarchy in the class of the usual commuting functions.

5 The superbosonic approach.

We now assume that the components of the vector supermultiplet F and G are superbosons and could be expressed as

$$F_i = f_i^1 + \theta_1 \zeta_i^1 + \theta_2 \zeta_i^2 + \theta_2 \theta_1 f_i^2, \tag{49}$$

$$G_i = g_i^1 + \theta_1 \eta_i^1 + \theta_2 \eta_i^2 + \theta_2 \theta_1 g_i^2, \tag{50}$$

where ζ_i^j and η_i^j are Grassmann valued functions while f_i^j, g_i^j are usual commuting functions. In order to find the proper Lax operator in this case we assume the following gradation on the functions

$$\begin{aligned} &\deg\left(f_i^1\right) = 0\,, \qquad \deg\left(\zeta_i^j\right) = 0.5\,, \qquad \deg\left(f_i^2\right) = 1, \\ &\deg\left(g_i^1\right) = 1\,, \qquad \deg\left(\eta_i^j\right) = 1.5\,, \qquad \deg\left(g_i^2\right) = 2,. \end{aligned} \tag{51}$$

Notice that it is possible also to assume the symmetrical gradation in which we replace $f \to g$, $\zeta \to \eta$ but we will not consider such possibility because we obtain the same information as in the considered case.

We postulate the Lax operator exactly as in the formula (22) and interestingly in this case we obtain the same flows, where now in contrast to eq. (23) F and G are superbosons. Therefore, they have different expansions in the components. Let us consider more carefully two particular cases ($k = 1$) of these flows. The second flow is

$$\frac{d}{dt}F = F_{xx} - G^2 F^3 + 2F\left(\mathcal{D}_1 \mathcal{D}_2 G F\right), \tag{52}$$

$$\frac{d}{dt}G = G_{xx} - G^3 F^2 - 2G\left(\mathcal{D}_1 \mathcal{D}_2 G F\right). \tag{53}$$

It is the extended supersymmetric Nonlinear Schrödinger Equation considered in [48]. The third flow is

$$\frac{d}{dt}F = F_{xxx} + 3F_x\left(\mathcal{D}_1 \mathcal{D}_2 G F\right) + 3F\left(\mathcal{D}_1 \mathcal{D}_2 G F_x\right) - 3F^2 G^2 F_x, \tag{54}$$

$$\frac{d}{dt}G = G_{xxx} + 3G_x\left(\mathcal{D}_1 \mathcal{D}_2 G F\right) + 3G\left(\mathcal{D}_1 \mathcal{D}_2 G_x F\right) - 3F^2 G^2 G_x. \tag{55}$$

From the last equation it follows that for $F = -1$, our equations reduces to the supersymmetric Korteweg - de Vries. As it is known there are three different generalization of the extended supersymmetric KdV equation which have the Lax representation [27–29,42,46] and this can be compactly written down as

$$\frac{d}{dt}G = \left(-G_{xx} + 3G\left(\mathcal{D}_1\mathcal{D}_2G\right) + \frac{1}{2}(\alpha-1)\left(\mathcal{D}_1\mathcal{D}_2G^2\right) + \alpha G^3\right)_x \tag{56}$$

Here, α is just a free parameter which enumerates these three different cases. Our case corresponds to $\alpha = 1$, after rescaling the time and transforming G into $-G$. In the paper [46] the present author considered the nonstandard Lax representations for this equation. Here, as the byproduct of our analysis we obtained the usual Lax representation for this equation which could be connected with the extended supersymmetric AKNS approach. Indeed, our Lax operator in this case takes the form

$$L = \partial - \partial^{-1}\mathcal{D}_1\mathcal{D}_2G \tag{57}$$

with the following flow

$$L_t = \left[\left(L^3\right)_+, L\right] . \tag{58}$$

Unfortunately, similarly to the superfermionic case considered in the previous chapter we have not found the bihamiltonian structure of this equation.

6 The superfermionic and superbosonic approach.

We are now able to consider the mixed approaches to the construction of the SUSY multicomponent KP hierarchy. Therefore we consider now the following SUSY lax operator

$$L = \partial + \sum_{i=1}^{k} F_i\partial^{-1}D_1D_2G_i + \sum_{j=1}^{m} B_i\partial^{-1}D_1D_2C_i, \tag{59}$$

where now F and G are vector superfermions with the expansions (20- 21) while B and C are superbosons with the following expansions (47-48). Using

the same technique as in the previous sections we computed the second and third flows but the final formulas are complicated. Hence we present the second flow only which can be written down as

$$\frac{d}{dt}F_i = F_{ixx} - F_i Z + 2\sum_{l=1}^{k} F_l\left(\mathcal{D}_1\mathcal{D}_2 G_l F_i\right) + 2\sum_{j=1}^{m} B_j\left(\mathcal{D}_1\mathcal{D}_2 C_j F_i\right), \tag{60}$$

$$\frac{d}{dt}B_j = B_{jxx} - B_j Z + 2\sum_{l=1}^{k} F_l\left(\mathcal{D}_1\mathcal{D}_2 G_l B_j\right) + 2\sum_{s=1}^{m} B_s\left(\mathcal{D}_1\mathcal{D}_2 C_s B_j\right), \tag{61}$$

$$\frac{d}{dt}G_i = -G_{ixx} + G_i Z - 2\sum_{l=1}^{k} G_l\left(\mathcal{D}_1\mathcal{D}_2 G_i F_l\right) - 2\sum_{j=1}^{m} C_j\left(\mathcal{D}_1\mathcal{D}_2 G_i B_j\right), \tag{62}$$

$$\frac{d}{dt}C_j = -C_{jxx} + C_j Z - 2\sum_{l=1}^{k} G_l\left(\mathcal{D}_1\mathcal{D}_2 C_j F_l\right) - 2\sum_{s=1}^{m} C_s\left(\mathcal{D}_1\mathcal{D}_2 C_j B_s\right). \tag{63}$$

where

$$Z = \left(\sum_{l=1}^{k} F_i G_i + \sum_{j=1}^{m} B_j C_j\right)^2 \tag{64}$$

As we see, the last system of equations describes the huge class of interacting fields. In some sense, it describes the interaction of the superfermions with the superbosons.

7 Concluding remarks.

We have constructed the extended supersymmetric version of the multicomponent K-P hierarchy in three different ways. We obtained a new class of integrable equation for which we were able to constructed the Lax operator and showed that they are Hamiltonian equations. Moreover, due to the existence of the Lax operator, we obtained the infinite number of conserved currents for our generalizations. Unfortunately we could not prove that these currents are in involution.

In the soliton theory, in order to prove the involution of the conserved currents, we utilize the recursion operator. Magri [53] has shown that such

recursion operator could be constructed if we know the bihamiltonian structure. However in our case we could not find such bihamiltonian structure. It does not mean that our system does not possess the recursion operator. The excellent example of the situation, where we do not known the bihamiltonian structure, but we known the recursion operator is the Burgers equation [55]. Therefore, it seems reasonable that if we wish to find the recursion operator for our supersymmetric generalizations we should try, to follow the Burgers approach.

References

[1] L. Faddeev and L. Takhtajan: "Hamiltonian Methods in the Theory of Solitons ", Springer (1987); A. Das: "Integrable Models", World Sci. (1989); M. Ablowitz and H. Segur: "Solitons and the Inverse Scattering Transform", SIAM Philadelphia (1981).

[2] A. Polyakov: in "Fields, Strings and critical Phenomena", ed. E. Brezin and J. Zinn-Justin, North Holland (1989).

[3] D. Gross and A. Migdal Nucl. Phys. **B340**, 333 (1990); E. Brezin and V. Kazakov Phys. Lett., **236B**, 144 (1990).

[4] S. Manakov, S. Novikov, L. Pitaevski and V. Zakharov: "Soliton Theory: The Inverse Problem", Nauka, Moscow (1980).

[5] L.A. Dickey: "Soliton Equations and Hamiltonian Systems", World Scientific, Singapore (1991).

[6] E. Date, M. Jimbo, M. Kashiwara and T. Miwa, in: "Nonlinear Integrable Systems - Classical and Quantum Theory", ed. by M. Jimbo and T. Miwa, World Scientific, Singapore 1983, p. 39.

[7] M. Jimbo and T. Miwa, Publ. RIMS, Kyoto Univ. **19**, 943 (1983).

[8] Y. Ohta, J. Satsuma, D. Takahashi, and T. Tokihiro,
Prog. Theor. Phys. Suppl. **94**, 219 (1988).

[9] B.G. Konopelchenko and W. Strampp, Inverse Problems **7**, L17 (1991).

[10] B.G. Konopelchenko, J. Sidorenko and W. Strampp, Phys. Lett., **A 157**, 17 (1991).

[11] J. Sidorenko and W. Strampp, Inverse Problems **7**, L37 (1991).

[12] Y. Cheng and Y. Li, Phys. Lett., **A157**, 22 (1991).

[13] Y. Cheng, J. Math. Phys., **33**, 3774 (1992).

[14] J. Sidorenko and W. Strampp, J. Math. Phys. **34**, 1429 (1993).

[15] W. Oevel and W. Strampp, Commun. Math. Phys. **157**, 51 (1993).

[16] N. Yajima and M. Oikawa, Progr. Theor. Phys. **56**, 1719 (1976).

[17] J. Wess and J. Bagger: "Supersymmetry and Supergravity", Princeton, NJ (1982);

[18] S. Ferrara and J.G. Taylor: "Introduction to Supergravity" (in russian) Moscow 1985.

[19] B. Kupershmidt: "Elements of Superintegrable Systems", Kluwer 1987.

[20] M. Chaichian, P. Kulish, Phys. Lett. **18B**, 413, (1978).

[21] R. D'Auria and S. Sciuto, Nucl. Phys. **B171**, 189, (1980).

[22] M. Gurses and O. Oguz, Phys. Lett. **108A**, 437, (1985).

[23] Y. Manin and R. Radul, Commun. Math. Phys. **98**, 65 (1985).

[24] C. Morosi and L. Pizzochero, Commun. Math. Phys. **158**, 267 (1993).

[25] C. Morosi and L. Pizzochero, J. Math. Phys., **35**, 2397, (1994).

[26] C. Morosi and L. Pizzochero: "A Fully Supersymmetric AKNS Theory ", preprint of Dipartimento di Matematica, Politecnico di Milano. April 1994, to appeare in Commun. Math. Phys. (1995).

[27] P. Mathieu, J. Math. Phys., **29**, 2499, (1988).

[28] C.A. Laberge and P. Mathieu, Phys. Lett. **215B**, 718, (1988).

[29] P. Labelle and P. Mathieu, J. Math. Phys., **32**, 923, (1991).

[30] M. Chaichian and J. Lukierski, Phys. Lett., **183B**, 169, (1987).

[31] T. Inami and H. Kanno, Commun. Math. Phys. **136**, 519 (1991).

[32] S.K. Nam, Intern. J. Mod. Phys., **4**, 4083, (1989).

[33] K. Hiutu and D. Nemeschansky, Mod. Phys. Lett. **A6**, 3179, (1991).

[34] C.M. Yung, Phys. Lett. **309B**, 75, (1993).

[35] E. Ivanov and S. Krivons, Phys. Lett. **291B**, 63, (1992).

[36] P. Kulish, Lett. Math. Phys. **10**, 87, (1985).

[37] G.H.M. Roelofs and P.H.M. Kersten, Journ. Math. Phys. **33**, 2185 (1992).

[38] J.C. Brunelli and A. Das, Journ. Math. Phys. **36**, 268, (1995).

[39] F. Toppan, Int. Journ. Mod. Phys. **A10**, 895, (1995).

[40] S. Krivonos and A. Sorin: "The minimal N=2 superextension of the NLS equation", hep-th/9504084 to appeare in Phys. lett. B.

[41] S. Krivonos, A. Sorin and F. Toppan: "On the Super- NLS Equation and its Relation with N=2 Super-KdV within Coset Approach", hep-th/9504138.

[42] W. Oevel and Z. Popowicz, Commun. Math. Phys. **139**, 441, (1991).

[43] Z. Popowicz, J. Phys. A:Math. Gen. **19**, 1495, (1986).

[44] Z. Popowicz, J. Math. A:Math. Gen. **23**, 1127, (1990).

[45] Z. Popowicz, Phys. Lett. **319B**, 478, (1993).

[46] Z. Popowicz, Phys. Lett. **174A**, 411, (1993).

[47] Z. Popowicz, Int. Jour. Mod. Phys. **9**, 2001, (1994).

[48] Z. Popowicz, Phys. Lett. **194A**, 375, (1994).

[49] S. Manakov, Sov. Phys. JEPT **38**, 248 (1974).

[50] V.K. Melnikv, Commun. Math. Phys. **112**, 639, (1987).

[51] Q.P. Liu: "Hamiltonian Structure of Multicomponent Constrained KP Hierarchy", hep-th/9502076.

[52] H. Aratyn, J.F. Gomes and A.H. Zimerman: "Affine Lie Algeraic Origin of Constrained KP Hierarchies", hep-th/9408104 to appeare in Journ. of Math. Phys.

[53] F. Magri, J. Math. Phys. **19**, 1156 (1978).

[54] M. Olshanetsky, Commun. Math. Phys. **30** (1975) 1975.

[55] W. Oevel and W. Strampp, Commun. Math. Phys. **157**, 51 (1983).